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A Polya-Hilbert operator for automorphic L-functions

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INTRODUCTION

In this paper it is shown that to any automorphic L-function there is a natural operator D (Polya-Hilbert operator) whose spectrum coincides with the set of zeros of the L-function along the critical line $\mathrm{Re}(s) = \frac{1}{2}$. This generalizes the first part of A. Connes's recent paper [4]. As a by-product a new proof of the meromorphicity of automorphic L-functions is established which gives new insights in the analytic behaviour of the L-functions.

According to the Langlands philosophy, the automorphic L-functions for GL_n should be the building blocks of all arithmetic L-functions, i.e. any arithmetic L-function should be expressible as a rational function in automorphic L-functions. Besides arithmetic also geometry gives rise to L-functions or zeta functions, like the zeta function of a graph [1], the zeta function of a diffeomorphism [10], the zeta function of a finite variety [6], or the zeta function of a flow [13]. In geometry one often has Lefschetz formulas relating the zeta function to the spectrum of a generating operator like the Frobenius one in the case of a finite variety or the infinitesimal generator in case of a flow [5]. Sometimes these formulas also yield essential steps in the proof of the corresponding version of the Riemann hypothesis. One would thus like to find the geometry of arithmetic L-functions. This could well be a 'noncommutative' geometry. For example, to express the Riemann zeta function as a zeta function of a flow one should give a natural flow having the set $\{\log(p) | p \text{ prime}\}$ as prime orbit length set. Such a flow indeed can be constructed on the (highly irregular) space

$\mathbb{R}/\mathbb{Q}^\times$, where the multiplicative group $\mathbb{Q}^\times$ acts by multiplication. The flow then is given by $\phi_t(x) = e^t x$. The space $Y = \mathbb{R}/\mathbb{Q}^\times$ is not Hausdorff but it can be embedded into the space $X = \mathbb{A}/\mathbb{Q}^\times$, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Then Y is a degenerate boundary component of the space X, which carries a rich structure. It is precisely this space on which A. Connes [4] constructs a natural space of functions (a Sobolev space), with an action of the flow such that the infinitesimal generator of this action is the Polya-Hilbert operator for the Riemann zeta function. In the present paper the ideas of A. Connes are extended to cover the case of automorphic L-functions for the group GL_n.

The paper is organized as follows. In the first section we merely set up the stage. For the sake of fixing notations we repeat in the second section a well known computation expressing the local L-factors as traces of elements of certain natural operators. Section 3 is the heart of the paper. We first show that the invariant sum of a Schwartz-Bruhat function is rapidly decreasing if the Schwartz-Bruhat function and its Fourier transform annihilate lower rank elements. This is the key lemma (3.1) which makes the analytic continuation particularly easy. We then show that to each automorphic representation π and each pair of distinct unramified places w_1, w_2 there is a Schwartz-Bruhat function f of the sort given above and a vector φ_π of π such that the pairing of f with φ_π times the unramified character $|x|^{s-(1/2)}$ equals

$$L_{w_1,w_2}(\pi, s) = \frac{L(\pi, s)}{L(\tilde{\pi}_{w_1}, 1-s)L(\pi_{w_2}, s)}.$$

This, together with the key lemma, in particular implies that this quotient is entire (Theorem 3.4). We then keep f and φ_π fixed and find that the generator D of a natural flow at ∞ has discrete spectrum on the orthogonal space of $f \otimes \varphi_\pi$ in the δ-Sobolev space for any $\delta > 0$. This orthogonal space consists all of distributions of order $< (\delta - 1)/2$ which are supported on the zero set of $L_{w_1,w_2}(\pi, (1/2) + it)$ and the spectrum of D on that space coincides with the set of zeros of $L_{w_1,w_2}(\pi, s)$ on the critical line (Theorem 3.2). In the last section we also give a version without the parameter $\delta > 0$. Then however we are no longer dealing with operators on Hilbert spaces but on more general topological vector spaces (Theorem 4.1).

The present paper grew out of an attempt to understand Connes' ideas by generalizing them to GL_n. For cuspidal L-functions the first result is in [7] and the second result has been achieved independently by C. Soulé [14].

I thank A. Connes for his interest in the present work.

1. SCHWARTZ-BRUHAT FUNCTIONS AND HAAR MEASURES

In this section we give some basic notations and facts needed in the sequel. For further details the reader is referred to [18]. Let k be a number field with ring of integers $\mathcal{O}$ and let M be a simple k-algebra with center k of rank n, i.e. $\dim_k(M) = n^2$. The standard example will be the algebra of $n \times n$ matrices $\mathrm{Mat}_n(k)$.

Choose a maximal $\mathcal{O}$-order $M(\mathcal{O})$ in M. For any commutative ring extension R of $\mathcal{O}$ let $M(R) = M(\mathcal{O}) \otimes_{\mathcal{O}} R$ and let G be the k-group of invertible elements, i.e. $G(R) = M(R)^{\times}$. For almost all places v of k we have $M(k_v) \cong \mathrm{Mat}_n(k_v)$, where k_v is the completion of k at v. In this case we say that M *splits at* v. At any splitting place v we assume an isomorphism $M(k_v) \to \mathrm{Mat}_n(k_v)$ fixed. We may and do assume the isomorphisms chosen compatibly in that the isomorphism comes by tensoring from an isomorphism $M(\mathcal{O}_v) \to \mathrm{Mat}_n(\mathcal{O}_v)$.

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{A}_\infty$ be the ring of adeles of k where $\mathbb{A}_f$ is the finite and $\mathbb{A}_\infty$ the infinite adeles. Let dx denote the additive Haar-measure on $\mathbb{A}$ given by $dx = \otimes_v dx_v$, the product being extended over all places of k, where at a finite place v we normalize the Haar measure so that for the valuation ring $\mathcal{O}_v \subset k_v$ we have $dx_v(\mathcal{O}_v) = 1$. At the infinite places we will normalize the measures in a way such that the lattice $k \subset \mathbb{A}$ has covolume 1, i.e. $\mathrm{vol}\,(\mathbb{A}/k) = 1$. We also write $dx = \otimes_v dx_v$ for an additive Haar-measure on $M(\mathbb{A})$, where dx_v is given component-wise by the above at each splitting place v.

For $a \in \mathbb{A}$ let $|a|$ be its modulus, so $dx(aA) = |a|dx(A)$. For $x \in M(\mathbb{A})$ let $|x| = |\det(x)|$, where $\det : M \to k$ is the reduced norm. Note that det equals the determinant over each field splitting M, which shows that the notation is consistent. Let $dx^\times = \otimes_v dx_v^\times$ be the Haar measure on $G(\mathbb{A})$ given by $dx_v^\times(G(\mathcal{O}_v)) = 1$ for v finite and $dx_v^\times = (dx_v/|x|_v^n)$ at the infinite places.

Let $\mathcal{S}(M(\mathbb{A}))$ be the space of Schwartz-Bruhat functions on $M(\mathbb{A})$, that is, any $f \in \mathcal{S}(M(\mathbb{A}))$ is a finite sum of functions of the form $f = \prod_v f_v$, where f_v is the characteristic function of the set $\mathcal{O}_v$ for almost all v and $f_v \in \mathcal{S}(M(k_v))$ at all places, where $\mathcal{S}(M(k_v))$ is the usual Schwartz-Bruhat space if v is infinite and is the space of locally constant functions of compact support if v is finite.

To define Fourier transforms we will fix an additive character ψ as follows. At first assume $k = \mathbb{Q}$, then $\psi = (\prod_p \psi_p)\psi_\infty$ with $\psi_p(\mathbb{Z}_p) = 1$, $\psi_p(p^{-n}) = e^{2\pi i/p^n}$, and $\psi_\infty(x) = e^{2\pi i x}$. For general k note that the trace map $Tr_{k/\mathbb{Q}} : k \to \mathbb{Q}$ induces a trace $Tr : \mathbb{A}_k \to \mathbb{A}_\mathbb{Q}$ and let $\psi_k := \psi_\mathbb{Q} \circ Tr$. The character ψ identifies $\mathbb{A}$ with its dual via the pairing $\langle x, y\rangle = \psi(xy)$. Note that ψ is chosen in a way that the lattice $k \subset \mathbb{A}$ is its own dual, i.e.

$$\langle x, y\rangle \;=\; 1 \;\; \forall y \in k \quad \Leftrightarrow \quad x \in k.$$

For $f \in \mathcal{S}(\mathbb{A})$ its Fourier transform is defined by

$$\hat{f}(x) \;=\; \int_{\mathbb{A}} f(y)\psi(xy)dy.$$

We lift these notions to $M(\mathbb{A})$. Let $\psi_M : M(\mathbb{A}) \to \mathbb{C}$ be defined by $\psi_M(x) = \psi(\mathrm{tr}_{M/k}(x))$. Then ψ_M sets $M(\mathbb{A})$ in self duality and $M(k)$ is a self dual lattice. The Fourier transform for $f \in \mathcal{S}(M(\mathbb{A}))$ is

$$\hat{f}(x) \;=\; \int_{M(\mathbb{A})} f(y)\psi_M(xy)dy$$

The Fourier transform preserves $\mathcal{S}(M(\mathbb{A}))$ and extends to an L^2-isometry with $\hat{\hat{f}}(x) = f(-x)$.

For any k-algebra R let $Q(R)$ be the set of all $m \in M(R)$ with $\det(m) = 0$. Note that for a field F we have $Q(F) = M(F) - G(F)$. Let $\mathcal{S}_0 = \mathcal{S}(M(\mathbb{A}))_0$ be the space of all $f \in \mathcal{S}(M(\mathbb{A}))$ such that f and $\hat{f}$ send $Q(\mathbb{A})$ to zero. Examples of these are easily given. Suppose $f = \prod_v f_v$ and there are two places $v \neq w$ with $\operatorname{supp} f_v \subset G(k_v)$ and $\operatorname{supp} \hat{f}_w \subset G(k_w)$ then f lies in $\mathcal{S}_0$.

2. LOCAL FACTORS

Let K be a nonarchimedean local field with ring of integers $\mathcal{O}$. We assume $K \supset k$ and M splits over K. So in this section read $M(K) = \mathrm{Mat}_n(K)$ and $G(K) = \mathrm{GL}_n(K)$. Let ϖ be a uniformizing element and let $|.| : K \to \mathbb{R}_{\geq 0}$ be the absolute value normalized so that $|\varpi| = q^{-1}$, where q is the number of elements of the residue class field $\mathcal{O}/\varpi\mathcal{O}$.

Let π be a representation of $G(K)$ on a complex vector space V, that is, π a group homomorphism from $G(K)$ to $GL(V)$. Then π is called *smooth* if for every $v \in V$ the stabilizer in $G(K)$ contains an open subgroup. If π is smooth then it is called *admissible* if $\dim V_\pi^U < \infty$ for any open subgroup U of $G(K)$, where V_π^U is the set of vectors fixed by any element of U. Fix an irreducible admissible representation π.

Suppose π is a *class one representation*, i.e. π has nonzero fixed vectors under the maximal compact subgroup $G(\mathcal{O})$ of $G(K)$. It then follows that the space of $G(\mathcal{O})$-fixed vectors is one dimensional. Any nonzero vector in this space will be called a *class one vector*. The *Hecke algebra* $\mathcal{H} = \mathcal{H}(G(K), G(\mathcal{O}))$ is the convolution algebra of all compactly supported functions $f : G(K) \to \mathbb{C}$ which are bi-invariant under $G(\mathcal{O})$, that is $f(xk) = f(kx) = f(x)$ for all $k \in G(\mathcal{O})$. By integration the algebra $\mathcal{H}$ acts on V_π and it preserves the space of class one vectors. Since the latter is one dimensional, $\mathcal{H}$ acts on it by a character χ_π. Further for any $f \in \mathcal{H}$ it holds that $\pi(f)$ is of rank one, therefore has a well defined trace and we have $\operatorname{tr} \pi(f) = \chi_\pi(f)$.

Let $A \subset G(K)$ be the subgroup of diagonal elements and $N \subset G(K)$ be the subgroup of upper triangular matrices with ones on the diagonal. Then $B = AN$ is the Borel subgroup of upper triangular matrices. For $a = \operatorname{diag}(a_1, \ldots, a_n) \in A$ let

$$\delta(a) := |\det(a|\mathrm{Lie}\, N)| = |a_1|^{n-1}|a_2|^{n-3} \ldots |a_n|^{-(n-1)}$$

the modulus of B. Let dn be the Haar measure on N normalized so that $\operatorname{vol}(N \cap G(\mathcal{O})) = 1$. The *Satake transform* (see [3], p. 146)

$$Sf(a) \ := \ \delta(a)^{\frac{1}{2}} \int_N f(an)dn$$

gives an isomorphism $\mathcal{H} \to \mathbb{C}[\Lambda]^W$, where $\Lambda = A/A \cap G(\mathcal{O})$ and $\mathbb{C}[\Lambda]$ is the convolution algebra (group algebra) of all finitely supported functions on Λ. Finally W is the Weyl group of all permutations of the diagonal entries of A. The set of characters $\mathrm{Hom}(\Lambda, \mathbb{C}^*)$ forms the complex points of a torus $T \subset \mathrm{GL}_n(\mathbb{C})$. The Hecke algebra $\mathcal{H}$ acts on the one dimensional space $V_\pi^{G(\mathcal{O})}$ through a character χ_π which by the Satake isomorphism is given by an element of T, also denoted χ_π. The *local L-factor* of π (and the standard representation of the Langlands dual group $GL_n(\mathbb{C})$) is defined by

$$L(\pi) = \det(1 - \chi_\pi)^{-1}.$$

We will need to make this more explicit. Let $\varpi_j = \mathrm{diag}(1, \dots, \varpi, \dots, 1)$, the ϖ on the j-th position. The Satake isomorphism gives a bijection

$$\mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C}^*)/W$$

and using these terms the local factor is given by

$$\begin{aligned} L(\pi)^{-1} &= \det(1 - \chi_\pi) \\ &= \prod_j (1 - \chi_\pi(\varpi_j)). \end{aligned}$$

Let $e \in \mathbb{C}[\Lambda]^W$ be given by $e = \prod_j (1 - \varpi_j)$. Then there exists a unique $f \in \mathcal{H}$ with $Sf = e$ and for this f we have

$$\mathrm{tr}\, \pi(f) = L(\pi)^{-1}.$$

But also $L(\pi)$ itself rather then its inverse can sometimes be given as a trace. Let $|\pi| := \max_j(|\chi_\pi(\varpi_j)|)$. The characteristic function $\mathbf{1}_{M(\mathcal{O})}$ restricted to $G(K)$ does not lie in $\mathcal{H}$ since it is not compactly supported but it can be written as an infinite sum of elements of $\mathcal{H}$. The same applies to the function $\mathbf{1}_{M(\mathcal{O})}|x|^{(n-1)/2}$.

Proposition 2.1. *If* $|\pi| < 1$ *then* $\pi(\mathbf{1}_{M(\mathcal{O})}|x|^{(n-1)/2})$ *exists and is of trace class with*

$$\mathrm{tr}\, \pi(\mathbf{1}_{M(\mathcal{O})}|x|^{\frac{n-1}{2}}) = L(\pi).$$

Proof. We compute the Satake transform of $f(x) = \mathbf{1}_{M(\mathcal{O})}(x)|x|^{(n-1)/2}$. At first note that $f(an) \neq 0$ implies $a \in A \cap M(\mathcal{O})$, that is $|a_j| \leq 1$ for $j = 1, \dots, n$. For such an a we substitute $y_{j,k}k = a_j n_{j,k}$, where $n_{j,k}$ is the corresponding entry of n. Since $\int_N f(n)dn = 1$ this gives

$$\int_N f(an)dn = |a|^{\frac{n-1}{2}} |a_1|^{-(n-1)} |a_2|^{-(n-2)} \dots |a_{n-1}|^{-1}.$$

And so

$$\begin{aligned} Sf(a) &= \delta(a)^{\frac{1}{2}} \int_N f(an)dn \\ &= |a|^{\frac{n-1}{2}} |a_1 a_2 \dots a_n|^{-\frac{n-1}{2}} \\ &= 1. \end{aligned}$$

So that $Sf = \mathbf{1}_{A \cap M(\mathcal{O})}$.

The condition $|\pi| < 1$ implies that

$$
\begin{aligned}
L(\pi) &= \det(1-\chi_\pi)^{-1}\\
&= \prod_j (1-\chi_\pi(\varpi_j))^{-1}\\
&= \prod_j \sum_{k=0}^{\infty} \chi_\pi(\varpi_j)^k\\
&= \chi_\pi\left(\prod_j \sum_{k=0}^{\infty} \varpi_j^k\right)\\
&= \chi_\pi(\mathbf{1}_{A\cap M(\mathcal{O})}). \quad \square
\end{aligned}
$$

The unramified character $g \mapsto |g|^s$ for some $s \in \mathbb{C}$ is a class one admissible representation of G and so is $\pi_s = |.|^s\pi : g \mapsto |g|^s\pi(g)$. We compute

$$
|\pi_s| = q^{-\mathrm{Re}(s)}|\pi|,
$$

so that we have the

Corollary 2.2. *For any irreducible admissible π and $s \in \mathbb{C}$ with $|\pi| < q^{\mathrm{Re}(s)}$ it holds*

$$
\begin{aligned}
L(\pi, s) &:= L(\pi_s)\\
&= \operatorname{tr} |.|^s\pi\left(\mathbf{1}_{M(\mathcal{O})}(x)|x|^{\frac{n-1}{2}}\right)\\
&= \operatorname{tr} \pi\left(\mathbf{1}_{M(\mathcal{O})}(x)|x|^{s+\frac{n-1}{2}}\right).
\end{aligned}
$$

3. GLOBAL L-FUNCTIONS

For $f \in \mathcal{S}_0$ let $E(f) : G(\mathbb{A}) \to \mathbb{C}$ be defined by

$$
\begin{aligned}
E(f)(x) &= |x|^{\frac{n}{2}} \sum_{\gamma \in M(k)} f(\gamma x)\\
&= |x|^{\frac{n}{2}} \sum_{\gamma \in G(k)} f(\gamma x).
\end{aligned}
$$

Note that $f \in \mathcal{S}_0$ implies that f vanishes on $\gamma M(\mathbb{A})$ for any $\gamma \in M(k) - G(k)$, so that we can sum over $G(k)$ or $M(k)$. Let further

$$
\begin{aligned}
\hat{E}(f)(x) &= |x|^{\frac{n}{2}} \sum_{\gamma \in M(k)} f(x\gamma)\\
&= |x|^{\frac{n}{2}} \sum_{\gamma \in G(k)} f(x\gamma).
\end{aligned}
$$

Proposition 3.1. *For any $f \in \mathcal{S}(M(\mathbb{A}))_0$ the sums $E(f)(x)$, $\hat{E}(f)(x)$ converge locally uniformly in x with all derivatives. For $x \in G(\mathbb{A})$ we have the functional equation:*

$$
E(f)(x) \;=\; \hat{E}(\hat{f})(x^{-1}).
$$

Further for any $N \in \mathbb{N}$ there is a $C > 0$ such that

$$|E(f)(x)|, |\hat{E}(f)(x)| \leq C \min(|x|, \frac{1}{|x|})^N.$$

Proof. With f any derivative of f is again in $\mathcal{S}(M(\mathbb{A}))$ so we only need to check convergence for f itself. Now $M(k)$ is a lattice in $M(\mathbb{A})$ and so is $M(k)x$ for any $x \in G(\mathbb{A})$. This lattice further depends continuously on x, which gives the first claim.

For any lattice $\Gamma \subset M(\mathbb{A})$ and any $f \in \mathcal{S}(M(\mathbb{A}))$ the Poisson summation formula [18] says

$$\operatorname{covol}(\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{\eta \in \Gamma^\perp} \hat{f}(\eta),$$

where $\Gamma^\perp$ is the dual lattice and $\operatorname{covol}(\Gamma) = \operatorname{vol}(\Gamma \backslash M(\mathbb{A}))$ is the covolume of Γ. Let $x \in G(\mathbb{A})$, then $M(k)x$ still is a lattice in $M(\mathbb{A})$ and its dual is $x^{-1}M(k)$. Further, the covolume of $M(k)x$ is $|x|^n$ so that

$$|x|^n \sum_{\gamma \in M(k)} f(\gamma x) = \sum_{\gamma \in M(k)} \hat{f}(x^{-1}\gamma),$$

which proves the functional equation. Now it remains to control the growth of $E(f)$ and $\hat{E}(f)$. By the functional equation it suffices to control the growth when $|x|$ is large. Since the argument is symmetric we only give the $E(f)$ case. We may assume that f is of the form $f = f_{fin} \otimes f_\infty$ with $f_{fin} \in \mathcal{S}(M(\mathbb{A}_{fin}))$ and $f_\infty \in \mathcal{S}(M(\mathbb{A}_\infty))$. The function f_{fin} has compact support, say $C'_{fin} \subset M(\mathbb{A}_{fin})$. Shifting factors we may assume that $|f_{fin}| \leq \mathbf{1}_{C'_{fin}}$ and so we can estimate

$$\begin{aligned} |E(f)(x)| &\leq |x|^{\frac{n}{2}} \sum_{\gamma \in G(k)} |f(\gamma x)| \\ &\leq |x|^{\frac{n}{2}} \sum_{\gamma \in G(k) \cap C'_{fin} x_{fin}^{-1}} |f_\infty(\gamma x_\infty)|. \end{aligned}$$

Lemma 3.2. *There is a compact subset D_{fin} of $G(\mathbb{A}_{fin})$ containing the unit and satisfying $G(k)D_{fin} = G(\mathbb{A}_{fin})$.*

Proof. Assume first that M splits over an infinite place of k. Then by strong approximation there is a finite set $E \subset G(\mathbb{A}_{fin})$ such that $G(k)E$ is dense in $G(\mathbb{A}_{fin})$ so in this case we can choose D_{fin} to be an arbitrary compact neighborhood of the unit times E. In the remaining case that M does not split over any infinite place it follows that k is totally real and M is a Quaternion algebra. It especially follows that the group $G^1 = \ker(\det)$ is totally anisotropic over k, so $G^1(k)$ is discrete and cocompact in $G^1(\mathbb{A})$ (see [11]), and, since $G^1(\mathbb{A}_\infty)$ is compact it follows that $G(k)$ is discrete and cocompact in $G^1(\mathbb{A}_{fin})$. So there is a compact set $D^1_{fin} \subset G^1(\mathbb{A}_{fin})$ such that $G^1(\mathbb{A}_{fin}) = G^1(k)D^1_{fin}$. Now G is the product of G^1 and a split torus so again there is a finite set E such that $G(\mathbb{A}_{fin}) = G(k)D^1_{fin}E$. The lemma follows. $\square$

Fix D_{fin} as above and let C_{fin} be an arbitrary compact open additive subgroup of $M(\mathbb{A}_{fin})$ that contains the compact set $C'_{fin}(D_{fin})^{-1}$, then C_{fin} still contains the support of f_{fin}.

Back to $E(f)(x)$ let $\tilde{x}$ be in $G(k)$ such that $\tilde{x}_{fin}^{-1}$ lies in $D_{fin}x_{fin}^{-1}$. Replacing the sum over γ by the sum over $\gamma\tilde{x}$ we see that $|E(f)(x)|$ is less than or equal to

$$|x|^{\frac{q}{2}} \sum_{\gamma \in G(k) \cap C_{fin}} |f_\infty(\gamma \tilde{x}^{-1} x_\infty)|.$$

There is a constant $d > 0$ such that $|\tilde{x}^{-1}|_{fin} \leq d|x^{-1}|_{fin}$. By the product formula we have that $|\tilde{x}^{-1}|_\infty = |\tilde{x}|_{fin} \geq \frac{1}{d}|x|_{fin}$ so that $|\tilde{x}^{-1}x_\infty|_\infty \geq \frac{1}{d}|x|$, this implies that, in order to prove bounds for $|x| \to \infty$ we may as well write x instead of $\tilde{x}^{-1}x_\infty$, when now x is supposed to be in $M(\mathbb{A}_\infty)$. The sum is extended over a fixed lattice Σ in $M(\mathbb{A}_\infty)$ given by $\Sigma = M(k) \cap C_{fin}$ which then is intersected with $G(\mathbb{A}_\infty)$. For any $\gamma \in G(k) \cap C_{fin}$ the product formula says $|\gamma|_{fin}|\gamma|_\infty = 1$ and $|\gamma|_{fin} \leq c$ for some $c > 0$. Hence $|\gamma_\infty| \geq \frac{1}{c}$. So we have the following situation: $\Lambda \subset M_\infty = M(\mathbb{A}_\infty)$ is a subset of a lattice and $|\lambda| \geq \frac{1}{c}$ for all $\lambda \in \Lambda$. (In our case we have $\Lambda = G(k) \cap C_{fin}$.) We have to estimate the sum $\sum_{\lambda \in \Lambda} f(\lambda x)$ for a Schwartz-Bruhat function f and $|x| \geq 1$.

Lemma 3.3. *Let $\| \, . \, \|$ be a norm on the finite dimensional real vector space M_∞. Let m be a natural number bigger than n^2. Then the sum*

$$\sum_{\lambda \in \Lambda} \| \lambda x \|^{-m}$$

is bounded on the set of all $x \in M_\infty$ with $|x| = 1$.

Note that the lemma, once proven for one norm $\| \, . \, \|$, immediately follows for any norm, since on the finite dimensional space M_∞ all norms are equivalent.

This lemma indeed proves the proposition since for a m as in the lemma there is $c_1 > 0$ such that $|f(z)| \leq c_1 \| z \|^{-m}$. The lemma implies that the sum

$$\sum_{\lambda \in \Lambda} \| \lambda x \|^{-m} |x|^{\frac{m}{n}}$$

is bounded on G_∞. This implies that $\sum_{\lambda \in \Lambda} |f(\lambda x)| \leq c_2 |x|^{-\frac{m}{n}}$. Since m can vary this implies the proposition.

Proof of the Lemma. Since C_{fin} is open there is a compact open subgroup U_{fin} of $G(\mathbb{A}_{fin})$ such that $C_{fin} = C_{fin}U_{fin}$. Then $\Gamma = G(k) \cap U_{fin}$ is an arithmetic subgroup of $G_\infty = G(\mathbb{A}_\infty)$ such that $\Gamma \subset \Lambda$ and $\Lambda\Gamma = \Lambda$. Let $P = LAN = LNA$ be the Langlands decomposition of a minimal k-parabolic in G_∞. Let U_∞ be a maximal compact subgroup of G_∞. Then $G_\infty = PU_\infty$. Since $LN/\Gamma \cap LN$ is compact there is a compact set $F_1 \subset LN$ such that $G_\infty = \Gamma F_1 A U_\infty$. We may assume that the Weyl group $W = W(A, G_\infty)$ has representatives in $\Gamma \cap U_\infty$. Let A^+ be the positive Weyl chamber in A given by P then we infer that there is a compact set $F_2 \subset LN$ such that $G_\infty = \Gamma F_2 A^+ U_\infty$. For any $a \in A^+$ the set $a^{-1}F_2 a$ remains in a fixed compact set $F_3 \subset LN$, therefore $G_\infty = \Gamma A^+ F_3 U_\infty$. For any y in $F_3 U_\infty$ let $\| x \|_y = \| xy \|$ then $\| \, . \, \|_y$ is a norm varying continuously with y. Since M_∞ is finite dimensional there is $C > 0$ such that $\frac{1}{C} \| \,.\, \| \leq \| \,.\, \|_y \leq C \| \,.\, \|$ for all y in the compact set $F_3 U_\infty$. It therefore suffices to show that $\sum_{\Lambda \in \Lambda} \| \lambda a \|^{-N}$ remains bounded for $a \in A^+$, further, if $x = ay$ satisfies $|x| = 1$, where y varies in a compact set then there is $\mu > 0$ such that $|a| \geq \mu$. So we have to show that $\sum_{\Lambda \in \Lambda} \| \lambda a \|^{-m}$ remains bounded for $a \in A^+$ with $|a| \geq \mu$.

We will now specialize the norm $\| \,.\, \|$. Consider the action of A on M_∞ given by $a.z = za$. Let $M_\infty = \bigoplus_\alpha E_\alpha$ be the decomposition into eigenspaces. So $\alpha \in \mathrm{Hom}(A, \mathbb{R}_+^\times)$ and E_α is the set of all $x \in M_\infty$ with $a.x = \alpha(a)x$ for all $a \in A$. Let $\| \,.\, \|_\alpha$ be any norm on E_α and let $\| v \| = \max_\alpha \| v_\alpha \|_\alpha$. Now let $N_\infty = N(\mathbb{A}_\infty)$. By integrality Λ is bounded away from N_∞. So there is $\epsilon > 0$ such that $d(\lambda, N_\infty) > 2\epsilon$ for any $\lambda \in \Lambda$ and $d(\lambda, \lambda') > 2\epsilon$ for any $\lambda \neq \lambda'$ in Λ, where d denotes the distance with respect to $\| \,.\, \|$. To given $\epsilon > 0$ there is $\delta > 0$ with $|y| \geq \delta$ for all $y \in M_\infty$ with $d(y, \Lambda) < \epsilon$. Fix $\lambda \in \Lambda$ and $y \in M_\infty$ with $d(y, \lambda) < \epsilon$. Now Λ is the subset of a lattice L in M_∞ which is a direct sum $L = \bigoplus_\alpha L_\alpha$ of lattices $L_\alpha \subset E_\alpha$. To see this, recall that by the theory of associative algebras [12] we have $M(k) = \mathrm{Mat}_n(D)$ for some division algebra D over k. Then $A(k)$ equals $(k^*)^n$ embedded as the diagonal. Then the spaces E_α are just the columns in $\mathrm{Mat}_n(D)$ for $j = 1$ to l.

It follows that, if $l = \sum_\alpha l_\alpha \in L$ lies in Λ then $l_\alpha \neq 0$ for all α. This implies that we can choose ϵ so small that $\| y \|_\alpha \leq 2 \| \lambda \|_\alpha$ for any α. Therefore, for any $a \in A$ we have $\| ya \| \leq \max_\alpha \alpha(a) \| y \|_\alpha \leq 2 \max_\alpha \alpha(a) \| \lambda \|_\alpha = 2 \| \gamma a \|$. Let η be the volume of the ball $B_\epsilon(0)$ around zero of radius ϵ. We conclude

$$\begin{aligned}
\sum_{\gamma \in \Gamma} \| \gamma a \|^{-m} &\leq \frac{2^m}{\eta} \sum_{\gamma \in \Gamma} \int_{B_\epsilon(\gamma)} \| ya \|^{-m}\, dy \\
&\leq \frac{2^m}{\eta} \int_{|y| \geq \delta} \| ya \|^{-m}\, dy \\
&= \frac{2^m}{|a|\eta} \int_{|y| \geq |a|\delta} \| y \|^{-m}\, dy \\
&\leq \frac{2^m}{\mu\eta} \int_{|y| \geq \mu\delta} \| y \|^{-m}\, dy. \quad \square
\end{aligned}$$

Remark. As a look at the proof shows, the lemma already holds for any Schwartz-Bruhat function f such that in the definition of $E(f)$ and $\hat{E}(\hat{f})$ the sum may mutually be extended over $G(k)$ or $M(k)$.

Let π be an irreducible admissible representation of $G(\mathbb{A})$ then $\pi = \otimes_v \pi_v$ and π_v is of class one for almost all places v. Let S be a finite set of places such that M splits outside S and π_v is of class one outside S. Then S contains all archimedean places by definition. Define the (partial) *global L-function* of π as

$$L^S(\pi) = \prod_{v \notin S} L(\pi_v),$$

provided the product converges.

Now again $g \mapsto |g|^s$ for $s \in \mathbb{C}$ is admissible and so is $\pi_s : g \mapsto |g|^s \pi(g)$. Let $L(\pi, s) := L(\pi_s)$.

Lemma 3.4. *Suppose the representation π is unitary, then for* $\mathrm{Re}(s) > n$ *the product $L^S(\pi, s)$ converges.*

Proof. This follows from Proposition 6.12 in [7]. □

Let $G(\mathbb{A})^1$ be the kernel of the map $g \mapsto |g|$. The exact sequence

$$1 \to G(\mathbb{A})^1 \xrightarrow{i} G(\mathbb{A}) \to \mathbb{R}_+^\times \to 1$$

splits. Moreover, there are splittings which make $G(\mathbb{A})$ a direct product of $G(\mathbb{A})^1$ and $\mathbb{R}_+^\times$. For example fix a place $v_0 | \infty$ and let $s : \mathbb{R}_+^\times \to G(\mathbb{A})$ be given by $s(t)_v = \mathbf{1}$ if $v \neq v_0$ and $s(t)_{v_0} = t^{\frac{1}{n}} \mathbf{1}$ then the map $(i, s) : G(\mathbb{A})^1 \times \mathbb{R}_+^\times \to G(\mathbb{A})$ is an isomorphism. We will fix such a splitting from now on. For $x \in G(\mathbb{A})$ we will write $x = (x^1, |x|)$ in these coordinates.

The group $G(k)$ is a lattice in $G(\mathbb{A})^1$. A representation π of $G(\mathbb{A})^1$ is called *automorphic* if it occurs as a subrepresentation of the right regular representation on $L^2(G(k)\backslash G(\mathbb{A})^1)$. With respect to our fixed splitting we are able to lift π to a representation of $G(\mathbb{A})$ by $\pi(x^1, |x|) := \pi(x^1)$ and we will thus consider it as a representation of $G(\mathbb{A})$. Then π decomposes into an infinite tensor product $\pi = \otimes_v \pi_v$. We will write Z for the image of the splitting map s and we will thus identify $L^2(G(k)\backslash G(\mathbb{A})^1)$ to $L^2(ZG(k)\backslash G(\mathbb{A}))$. Write R for the representation of $G(\mathbb{A})$ on the latter. That is, for $\varphi \in L^2(ZG(k)\backslash G(\mathbb{A}))$ and $y \in G(\mathbb{A})$ we write $R(y)\varphi(x) = \varphi(xy)$.

At the first sight our definition of an automorphic representation seems too restrictive since we only allow central characters which are trivial on Z. To the L-function this however only means a shift in the argument. So everything we deduce for L-functions is perfectly general.

Now assume π is automorphic and fix a $G(\mathbb{A})^1$-homomorphism from V_π to $L^2(G(k)\backslash G(\mathbb{A})^1)$. Let $\varphi_\pi \in L^2(ZG(k)\backslash G(\mathbb{A}))$ be the image of some vector $\alpha = \otimes_v \alpha_v$ in $V_\pi = \otimes_v V_{\pi,v}$ such that α_v is a normalized class one vector at almost all places. Further assume that φ_π is smooth and $\varphi_\pi(1) \neq 0$. The latter can be achieved by replacing $\varphi_\pi(x)$ with $\varphi_\pi(xy)$ for a suitable y if necessary.

For any set of places S let $\mathbb{A}_S$ be the restricted product of k_v, $v \in S$ and let $G_S = G(\mathbb{A}_S)$. We consider G_S as a subgroup of $G(\mathbb{A})$. Further let $\mathbb{A}^S$ be the restricted product over all $v \notin S$.

Lemma 3.5. *Let $f \in \mathcal{S}(M(\mathbb{A}))_0$ be of the form $f_S \otimes f^S$ for a finite set of places S, where $f_S \in \mathcal{S}(M(\mathbb{A}_S))$ and $f^S = \prod_{v \notin S} \mathbf{1}_{M(\mathcal{O}_v)}$. Take S so large that M splits outside S and φ is of class one outside S, then for* $\mathrm{Re}(s) >> 0$ *we have*

$$\int_{G(k)\backslash G(\mathbb{A})} E(f)(x)\varphi_\pi(x)|x|^s d^*x \;=\; L^S(\pi, s+\tfrac{1}{2}) \int_{G_S} f(x)\varphi_\pi(x)|x|^{s+n/2} d^*x.$$

Proof. For $\mathrm{Re}(s) >> 0$ we compute that

$$\int_{G(k)\backslash G(\mathbb{A})} E(f)(x)\varphi_\pi(x)|x|^s d^*x$$

equals

$$\begin{aligned}\int_{G(\mathbb{A})} f(x)|x|^{\frac{g}{2}+s}\varphi_\pi(x)d^*x &= \int_{G(\mathbb{A})} f(x)|x|^{\frac{g}{2}+s}R(x)\varphi_\pi(1)d^*x \\ &= \prod_{v\neq S}\int_{G(k_v)} f_v(x)|x|^{\frac{g}{2}+s}R(x)\varphi_\pi(1)d^*x \\ &\qquad \times \int_{G_S} f(x)|x|^{\frac{g}{2}+s}\varphi_\pi(x)d^*x \\ &= \prod_{v\neq S} R(f_v(x)|x|^{\frac{g}{2}+s})\varphi_\pi(1) \\ &\qquad \times \int_{G_S} f(x)|x|^{\frac{g}{2}+s}\varphi_\pi(x)d^*x \\ &= \prod_{v\notin S} L(\pi_v, s+\frac{1}{2})\int_{G_S} f(x)|x|^{\frac{g}{2}+s}\varphi_\pi(x)d^*x.\end{aligned}$$

For the justification of this computation note that for $\mathrm{Re}\,(s) >> 0$ the integral over $G(\mathbb{A})$ converges absolutely. The last equality follows from Corollary 2.2 applied to all places not in S. □

Now recall from [7] ,Theorem 3.3 that there also are local L-factors $L(\pi_v, s)$ at the places in S so we can define

$$L_S(\pi, s) \;=\; \prod_{v\in S} L(\pi_v, s), \qquad L(\pi, s) = L_S(\pi, s)L^S(\pi, s).$$

Let $\epsilon(\pi, s)$ be the global ϵ-factor from p. 149 of [7].

Theorem 3.1. *Let π be an automorphic representation. Then $L(\pi, s)$ extends to a meromorphic function and satisfies the functional equation*

$$L(\pi, s) \;=\; \epsilon(\pi, s)L(\tilde{\pi}, 1-s),$$

where $\tilde{\pi}$ is the representation dual to π. Further $L(\pi, s)$ has at most finitely many poles which lie in $-(n-1)/2 \le \mathrm{Re}\,(s) \le (n+1)/2$. Finally, if w_1 and w_2 are distinct unramified places then

$$\frac{L(\pi, s)}{L(\tilde{\pi_{w_1}}, 1-s)L(\pi_{w_2}, s)}$$

is entire.

Remark. The first statement already is in Jacquet's paper [9]. The proof given here is more direct than Jacquet's and is more in the spirit of Tate's thesis. Jacquet essentially applies the methods of Tate to cuspidal representations first and than uses a decomposition result which allows to relate arbitrary automorphic L-functions to cuspidal ones. The present proof does it in one step.

Moreover the last statement of the theorem improves the corresponding statement in [9], where one only can say that there is a finite number of Euler factors which have to be inverted to get an entire function. The meromorphicity of the L-functions is also shown in [15]. Our result however is stronger.

Proof. In the notations of the last lemma let

$$\Delta_{S,\varphi_\pi,s}(f) := \int_{G_S} f(x)\varphi_\pi(x)|x|^{s+\frac{n}{2}}d^*x.$$

It follows from Lemma 3.5 and Proposition 3.1 that $L^S(\pi, s+\frac{1}{2})\Delta_{S,\varphi,s}(f)$ is an entire function in s.

Let w_1, w_2 be finite place outside S and let $S' = S \cup \{w_1, w_2\}$, then all we said also holds for S' instead of S. We may assume $\varphi_{\pi,w_1}(1) = \varphi_{\pi,w_2}(1) = 1$.

We need to fix f_{w_1} in a way to guarantee that $\hat{f}(Q(\mathbb{A})) = 0$. For this let f_{w_1} be the unique function in $\mathcal{S}(M(k_{w_1}))$ with

$$\hat{f}_{w_1} = q^{n^2}\mathbf{1}_{1+,\varpi M(\mathcal{O}_{w_1})},$$

then for any choice of f at the other places we have $\hat{f}(Q(\mathbb{A})) = 0$. Next we set $f_{w_2} = q^{n^2}\mathbf{1}_{\varpi M(\mathcal{O}_{w_2})}$, then also $f(Q(\mathbb{A})) = 0$, so $f \in \mathcal{S}_0$. Since φ_π is a class one vector at w_j we conclude that φ_{π,w_j} is constant on $1 + \varpi M(\mathcal{O}_{w_j})$ for $j = 1, 2$.

For $v \in S$ we choose local functions $f_{v,i}$ and $\varphi_{\pi,v,i}$ for $i = 1, \dots N_v$ so that

$$\sum_{i=1}^{N_v} \int_{G(k_v)} f_{v,i}(x)\varphi_{\pi,v,i}(x)|x|^{s+\frac{n}{2}}d^*x = L(\pi_v, s),$$

which is possible by [7] Theorem 3.3 (3) and Theorem 8.7. For any family $I = (i_v)_{v\in S}$ with $i_v \in \{1, \dots, N_v\}$ let

$$f_I = \prod_{v\notin S} f_v \prod_{v\in S} f_{v,i_v}$$

and

$$\varphi_{\pi,I} = \bigotimes_{v\notin S} \varphi_{\pi,v} \otimes \bigotimes_{v\in S} \varphi_{\pi,v,i_v},$$

then we get that

$$\sum_I \int_{G(k)\backslash G(\mathbb{A})} E(f_I)\varphi_{\pi,I}(x)|x|^s d^*x$$

equals

$$\sum_I \int_{G(\mathbb{A})} f_I(x)\varphi_{\pi,I}(x)|x|^{\frac{n}{2}+s}d^\times x$$

$$\prod_{v\in S}\left(\sum_{i=1}^{N_v} f_{v,i}(x)\varphi_{\pi,v,i}(x)|x|^{\frac{n}{2}+s}d^\times x\right) \times \prod_{v\notin S} \int_{G(k_v)} f_v(x)\varphi_{\pi,v}(x)|x|^{\frac{n}{2}+s}d^\times x$$

and this equals

$$L(\pi, s+\frac{1}{2})\frac{\Delta_{w_1,\varphi_{\pi,w_1},s}(f_{w_1})}{L(\pi_{w_1}, s+\frac{1}{2})}\frac{\Delta_{w_2,\varphi_{\pi,w_2},s}(f_{w_2})}{L(\pi_{w_2}, s+\frac{1}{2})}.$$

So this is an entire function in s. According to [7], Theorem 3.3(2), p. 30 we have

$$\Delta_{w_1,\varphi_\pi,s}(f_{w_1}) = \Xi_{w_1}(s+\frac{1}{2})L(\pi_{w_1}, s+\frac{1}{2}),$$

where $\Xi_{w_1}(s+\frac{1}{2})$ is an entire function. Let $\varphi_\pi^\vee(x) = \varphi_\pi(x^{-1})$. Then the local functional equation [7] Theorem 3.3 (4) shows

$$\Xi_{w_1}(s+\frac{1}{2}) = \pm\epsilon(\pi_{w_1}, \frac{1}{2}-s)\hat{\Xi}_{w_1}(\frac{1}{2}-s),$$

where

$$\begin{aligned}
\hat{\Xi}_{w_1}(\frac{1}{2}-s) &= \frac{1}{L(\tilde{\pi}_{w_1}, 2-s)}\Delta_{w_1,\varphi_\pi^\vee,-s}(\hat{f}) \\
&= \frac{1}{L(\tilde{\pi}_{w_1}, 2-s)}q^{n^2}\int\limits_{1+\varpi M(\mathcal{O}_{w_1})}\varphi_\pi^\vee(x)|x|^{-s+\frac{n}{2}}d^*x \\
&= \frac{1}{L(\tilde{\pi}_{w_1}, \frac{1}{2}-s)\mathrm{vol}\,(G(\mathcal{O}_{w_1}))},
\end{aligned}$$

so that

$$\Delta_{w_1,\varphi_\pi,s}(f_{w_1}) = \pm\frac{\epsilon(\pi_{w_1}, \frac{1}{2}-s)L(\pi_{w_1}, s+\frac{1}{2})}{L(\tilde{\pi}_{w_1}, \frac{1}{2}-s)\mathrm{vol}\,(G(\mathcal{O}_{w_1}))}. \quad \square$$

We have shown

Lemma 3.7. *Let w_1, w_2 be two finite places at which M splits and π is unramified. There is $f \in \mathcal{S}_0$ and $\varphi_\pi \in L^2(G(k)\backslash G(\mathbb{A}))$ such that*

$$\int\limits_{G(k)\backslash G(\mathbb{A})} E(f)(x)\varphi_\pi(x)|x|^{s-\frac{1}{2}}d^*x = \frac{L(\pi, s)}{L(\tilde{\pi}_{w_1}, 1-s)L(\pi_{w_2}, s)}.$$

Proof. This follows from above and Proposition 6.12 in [7]. $\square$

This implies that $(L(\pi,s)/L(\tilde{\pi}_{w_1}, 1-s)L(\pi_{w_2}, s))$ is entire. Now Theorem 3.3 in [7] also implies that the ϵ-factor is invertible, hence $(L(\pi,s)/L(\tilde{\pi}_{w_1}, 1-s)L(\pi_{w_2}, s))$ is entire. By Proposition 6.12 of [7] it now follows that the meromorphic function $L(\pi,s)$ is regular outside $-(n-1)/2 \le \mathrm{Re}\,(s) \le (n+1)/2$. To see that $L(\pi,s)$ only has finitely many poles recall that all these poles must be poles of $L(\tilde{\pi}_{w_1}, 1-s)$ or $L(\pi_{w_2}, s)$. These are rational functions in $q_{w_1}^{-s}$ and $q_{w_1}^{-s}$. Replacing w_1 and w_2 by places of different residue characteristic shows that these can only be finitely many.

By Proposition 3.1 we get that

$$\sum_I \int\limits_{G(k)\backslash G(\mathbb{A})} E(f_I)(x)\varphi_{\pi,I}(x)|x|^{s-\frac{1}{2}}d^*x$$

equals

$$\sum_I \int_{G(\mathbb{A})/G(k)} \hat{E}(\hat{f})\varphi^{\vee}_{\pi,I}(x)|x|^{\frac{1}{2}-s}d^*x.$$

We see that $\varphi^{\vee}_{\pi,I}$ is a vector of $L^2(G(\mathbb{A})/G(k))(\tilde{\pi})$ and our computations repeated mutatis mutandis for the right hand side imply

$$\frac{L(\pi,s)}{L(\tilde{\pi_{w_1}},1-s)L(\pi_{w_2},s)} = \epsilon(\pi,s)\frac{L(\tilde{\pi},1-s)}{L(\tilde{\pi_{w_1}},1-s)L(\pi_{w_2},s)}.$$

The theorem follows. □

For $\delta > 0$ let $L^2_\delta(G(k)\backslash G(\mathbb{A}))$ denote the space of measurable functions φ on $G(k)\backslash G(\mathbb{A})$ with

$$\int_{G(k)\backslash G(\mathbb{A})} |\varphi(x)|^2\left(1+(\log|x|)^2\right)^{\delta/2}d^*x < \infty$$

modulo nullfunctions. The sum E defines a linear map from $\mathcal{S}(M(\mathbb{A}))_0$ to $L^2_\delta(G(k)\backslash G(\mathbb{A}))$ The group $G(\mathbb{A})$ acts on both sides by right translation R and it is easy to see that for any $y \in G(\mathbb{A})$

$$ER(y) = |y|^{-n/2}R(y)E,$$

so that the image of E is an invariant subspace.

The pairing

$$(f,g) = \int_{G(k)\backslash G(\mathbb{A})} f(x)g(x)d^*x$$

identifies $L^2_{-\delta}(G(k)\backslash G(\mathbb{A}))$ to the dual of the space $L_\delta(G(k)\backslash G(\mathbb{A}))$.

The space $L^2_\delta(G(k)\backslash G(\mathbb{A}))$ can be viewed as Hilbert space tensor product

$$L^2(G(k)\backslash G(\mathbb{A})^1) \otimes L^2_\delta(\mathbb{R}),$$

where $L^2_\delta(\mathbb{R})$ is the Fourier transform of the δ-Sobolev space, i.e. the space of all functions f on $\mathbb{R}$ with $\int_{\mathbb{R}} |f(x)|^2(1+x^2)^{\delta/2}dx < \infty$.

Let $\tilde{\varphi} = \varphi_\pi \otimes \psi \in L^2_{-\delta}(G(k)\backslash G(\mathbb{A}))$, where φ_π is chosen as in Lemma 3.7. Let further $f \in \mathcal{S}_0$ be chosen as in Lemma 3.7, then $\tilde{\varphi}$ is orthogonal to $E(f)$ if and only if

$$\int_{G(k)\backslash G(\mathbb{A})} E(f)(x)\varphi_\pi(x)\psi(\log|x|)d^*x = 0.$$

Considering the function ψ as a distribution we may formally write

$$\psi(\log|x|) = \int_{\mathbb{R}} \hat{\psi}(t)|x|^{it}dt.$$

The above becomes

$$\int_{G(k)\backslash G(\mathbb{A})} \int_{\mathbb{R}} E(f)(x)\varphi_\pi(x)\hat{\psi}(t)|x|^{it}dtd^*x$$

which equals

$$\int_{\mathbb{R}} \frac{L(\pi,\frac{1}{2}+it)}{L(\tilde{\pi_{w_1}},\frac{1}{2}-it)L(\pi_{w_2},\frac{1}{2}+it)}\hat{\psi}(t)dt.$$

We can convolve the function $t \mapsto f(s(e^t)x)$ for $x \in G_S^1$ with an arbitrary Schwartz-Bruhat function in the variable t. Using the fact that Fourier transform turns convolution into pointwise product we find that to any compactly supported function α on $\mathbb{R}$ there is a test function $\phi_{f,\alpha}$ on G_S such that

$$\Delta_{S,\varphi_\pi,it}(\phi_{f,\alpha}) = \alpha(t)\Delta_{S,\varphi_\pi,it}(f).$$

We get that

$$\int_{\mathbb{R}} \frac{L(\pi,\frac{1}{2}+it)}{L(\tilde{\pi_{w_1}},\frac{1}{2}-it)L(\pi_{w_2},\frac{1}{2}+it)}\hat{\psi}(t)\alpha(t)dt = 0$$

for any compactly supported function α, hence

$$(1) \qquad \frac{L(\pi,\frac{1}{2}+it)}{L(\tilde{\pi_{w_1}},\frac{1}{2}-it)L(\pi_{w_2},\frac{1}{2}+it)}\hat{\psi}(t) = 0,$$

as a distribution in the variable $t \in \mathbb{R}$.

Now $\psi \in L^2_{-\delta}$ implies that its Fourier transform is a distribution of order $< (\delta - 1)/2$. For this recall that for $k \in \mathbb{N}$ the k-th derivative of the Dirac distribution $\delta^{(k)}(h) = h^{(k)}(0)$ has Fourier transform $(ix)^k$ which lies in $L^2_{-\delta}(\mathbb{R})$ if and only if $k < (\delta - 1)/2$. Therefore equation (1) is satisfied precisely for ψ being a linear combination of $\delta^{(k)}(t)$ where $k < (\delta - 1)/2$ and $\frac{1}{2} + it$ is a zero of $L(\pi)$ of order $> k$. We will interpret this as a spectral decomposition of the t-multiplication.

Let φ_π be as in Lemma 3.7. Define

$$L^2_\delta(G(k)\backslash G(\mathbb{A}))(\varphi_\pi) = \mathbb{C}\varphi_\pi \otimes L^2_\delta(Z) \subset L^2_\delta(G(k)\backslash G(\mathbb{A})).$$

Fix f, φ_π as in Lemma 3.7 and let $H \subset L^2_{-\delta}(G(k)\backslash G(\mathbb{A}))(\varphi_\pi)$ be the orthogonal space of the element $E(f) \in L^2_\delta(G(k)\backslash G(\mathbb{A}))$.

Our considerations lead to

Lemma 3.8. *The space H is spanned by functions of the form*

$$\varphi_\pi \otimes \psi \in L^2_{-\delta}(G(k)\backslash G(\mathbb{A}))(\bar{\varphi_\pi}),$$

where ψ is of the form

$$(i \log |x|)^k |x|^{it_0},$$

for t_0 a real zero of order m of the function

$$t \mapsto \frac{L(\pi,\frac{1}{2}+it)}{L(\tilde{\pi_{w_1}},\frac{1}{2}-it)L(\pi_{w_2},\frac{1}{2}+it)},$$

and $k < \min((\delta - 1)/2, m)$.

On H we have the operator D given by

$$D\xi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(R(s(e^{\epsilon})) - 1)\xi,$$

where $s : \mathbb{R}_+^\times \to G(\mathbb{A})$ is our fixed section. The domain dom D is the set of ξ for which the limit exists.

Lemma 3.9. *For $y \in G(\mathbb{A})$ we have for the operator norm:*

$$\| R(y) \|_\delta \leq 2^{\delta/4}(1 + (\log|y|)^2)^{\delta/4}.$$

Proof. Let $a, b \in \mathbb{R}$, then $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$ leads to

$$1 + (a+b)^2 \leq 2(1+a^2)(1+b^2).$$

Let $x, y \in G(\mathbb{A})$ and apply this inequality to $a = \log|x|$ and $b = \log|y^{-1}|$. Next apply the monotonous function $t \mapsto t^{\delta/2}$ to get

$$(1 + (\log|x| + \log|y^{-1}|)^2)^{\delta/2} \leq 2^{\delta/2}(1 + (\log|x|)^2)^{\delta/2}(1 + (\log|y^{-1}|)^2)^{\delta/2}.$$

Now let $y \in G(\mathbb{A})$ and $\varphi \in L^2_\delta(G(k)\backslash G(\mathbb{A}))$, then

$$\begin{aligned} \| R(y)\varphi \|_\delta^2 &= \int_{G(k)\backslash G(\mathbb{A})} |\varphi(xy)|^2(1 + (\log|x|)^2)^{\delta/2} d^*x \\ &= \int_{G(k)\backslash G(\mathbb{A})} |\varphi(x)|^2(1 + (\log|x| + \log|y^{-1}|)^2)^{\delta/2} d^*x \\ &\leq 2^{\delta/2}(1 + (\log|y|)^2)^{\delta/2} \int_{G(k)\backslash G(\mathbb{A})} |\varphi(x)|^2(1 + (\log|x|)^2)^{\delta/2} d^*x \\ &= 2^{\delta/2}(1 + (\log|y|)^2)^{\delta/2} \| \varphi \|_\delta^2 . \quad \square \end{aligned}$$

Lemma 3.10. *The operator D is closed and has purely imaginary spectrum. Its resolvent $Res_\lambda = (D - \lambda)^{-1}$ is given for* $\mathrm{Re}(\lambda) > 0$ *by*

$$\mathrm{Res}_\lambda = -\int_0^\infty R_\pi(s(e^t))e^{-\lambda t} dt$$

and for $\mathrm{Re}(\lambda) < 0$ *by*

$$\mathrm{Res}_\lambda = \int_0^\infty R_\pi(s(e^{-t}))e^{\lambda t} dt.$$

Proof. The closedness follows from the integral representation of Res_λ since the latter implies that the image of Res_λ lies in the domain of D. One proceeds as follows: Assume $v_n \to v$ and $Dv_n \to y$ and set

$$v' = \mathrm{Res}_\lambda y - \lambda \mathrm{Res}_\lambda v,$$

then v' lies in the domain of D and $(D - \lambda)v' = y - \lambda v$ and so $(D - \lambda)(v_n - v') \to 0$. Applying the continuous operator $(D - \lambda)^{-1}$ we get $v_n - v' \to 0$ and hence $v' = v$, so v lies in the domain of D and $Dv = y$.

To prove the integral representations recall that by definition D is the infinitesimal generator of the group $R_\pi(s(e^t))$, therefore $R_\pi(s(e^t)) = e^{tD}$ and Lemma 3.9 implies

$$\| e^{tD} \| \leq 2^{\delta/4}(1+t^2)^{\delta/4}.$$

Form this it follows that both integral converge in the range given. We compute for $\mathrm{Re}(\lambda) > 0$:

$$(D-\lambda)\left(-\int_0^\infty R_\pi(s(e^t))e^{-\lambda t}dt\right)$$

$$= -\int_0^\infty (D-\lambda)e^{t(D-\lambda)}dt = -e^{t(D-\lambda)}|_0^\infty = Id,$$

and similarly for $\mathrm{Re}(\lambda) < 0$. The lemma is proven. $\square$

Let t_0 be a zero of the function

$$\frac{L(\pi, \frac{1}{2}+it)}{L(\tilde{\pi_{w_1}}, \frac{1}{2}-it)L(\pi_{w_2}, \frac{1}{2}+it)}$$

and apply the operator $D = (d/dt)|_{t=0}R(e^t)$ to the function $\varphi_\pi \otimes \psi$, where ψ equals

$$(i \log|x|)^k |x|^{it_0}.$$

The result is φ_π tensored with

$$\begin{aligned}\frac{d}{dt}|_{t=0}(i\log|s(e^t)x|)^k|s(e^t)x|^{it_0} &= i^k\frac{d}{dt}|_{t=0}(t+\log|x|)^k e^{itt_0}|x|^{it_0}\\ &= i^k k(\log|x|)^{k-1}|x|^{it_0} + it_0(i\log|x|)^k|x|^{it_0}\end{aligned}$$

By virtue of Lemma 3.8 this implies that we have a basis of H consisting of generalized eigenvectors of D. Recall that a generalized eigenvector to the eigenvalue $\lambda \in \mathbb{C}$ is a vector $v \neq 0$ which satisfies $(D-\lambda)^n v = 0$ for some natural number n. We have shown:

Theorem 3.2. *For $\delta > 1$ the operator D has discrete spectrum in $i\mathbb{R}$ consisting of all $i\rho \in i\mathbb{R}$ such that $(1/2) + i\rho$ is a zero of $(L(\pi, s)/L(\tilde{\pi_{w_1}}), 1-s)L(\pi_{w_2}, s)$ for any two places w_1, w_2 where π is unramified. The generalized eigenspace at ρ has dimension $n(\rho)$, where $n(\rho)$ is the largest integer $\underset{\sim}{n} < (\delta+1)/2$ such that $n \leq$ multiplicity of the zero $(L(\pi, \frac{1}{2}+i\rho)/L(\pi_{w_1}, \frac{1}{2}-i\rho)L(\pi_{w_2}, \frac{1}{2}+i\rho))$.*

Remark. If the automorphic representation π is cuspidal the contents of Theorem 3.1 and Theorem 3.2 can be sharpened. In [7] it is shown that then $L(\pi, s)$ is entire. However in general not all automorphic L-functions are entire. Further in [14] the author proved that in Theorem 3.2 the quotient

$(L(\pi, s)/L(\tilde{\pi_{w_1}}, 1-s)L(\pi_{w_2}, s))$ can be replaced by $L(\pi, s)$ if simultaneously the map E is extended to $S(M(\mathbb{A}))$.

4. ELIMINATING δ

A major drawback of Theorem 3.2 is the dependence on the parameter $\delta > 0$. If one is willing to consider operators on other but Hilbert spaces this can be eliminated as follows.

Let $L(Z)$ be the space of functions of logarithmic growth, i.e. the space of smooth functions ψ on Z such that for any $N \in \mathbb{N}$ there is $C_N > 0$ with

$$|\psi(x| \leq C_N(1 + |\log|x||)^N.$$

Let

$$L = L^2(G(k)\backslash G(\mathbb{A})^1) \otimes L(Z)$$

and consider L as a subspace of the space of functions on $G(k)\backslash G(\mathbb{A})$. Let R denote the space of smooth functions on $G(k)\backslash G(\mathbb{A})$ of rapid decay, i.e. of functions F such that for any $N \in \mathbb{N}$ there is $d_N > 0$ such that

$$|F(x)| \leq d_N \min(|x|, \frac{1}{|x|})^N$$

for any $x \in G(k)\backslash G(\mathbb{A})$. There is a natural pairing

$$R \times L \to \mathbb{C}$$
$$(F, \psi) \mapsto \int_{G(k)\backslash G(\mathbb{A})} F(x)\psi(x)dx.$$

Let φ_π be as above and let

$$L(\varphi_\pi) = \mathbb{C}\varphi_\pi \otimes L(Z)$$

and let $L^\perp(\varphi_\pi)$ denote the orthogonal space in $L(\varphi_\pi)$ of the element $E(f)$, where f is as in Lemma 3.7. Then the operator $D = (d/dt)|_{t=0}R(s(e^t))$ acts on $L^\perp(\varphi_\pi)$ and the same reasoning as in the previous section gives

Theorem 4.1. *The space $L^\perp(\varphi_\pi)$ has a basis of generalized eigenvectors of D. The spectrum of D on $L^\perp(\varphi_\pi)$ coincides with the set of zeros ρ of the function*

$$\frac{L(\pi, s)}{L(\tilde{\pi_{w_1}}, 1-s)L(\pi_{w_2}, s)}$$

with $\mathrm{Re}(\rho) = \frac{1}{2}$. *The multiplicity of the eigenvalue ρ equals its order as a zero.*

REFERENCES

1. Bass, H. – The Ihara-Selberg zeta function for a tree lattice. Int. J. Math. 3, No. 6, 717-797 (1992).

2. Borel, A. – Automorphic L-functions. Automorphic forms, representations and L-functions, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. **33**, 2, 27-61 (1979).
3. Cartier, P. – Representations of p-adic groups: A survey. Automorphic forms, representations and L-functions, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. **33**, 1, 111-155 (1979).
4. Connes, A. – Trace formulas in noncommutative geometry and the zeros of the Riemann zeta function. Sel. Math., New Ser. **5**, No. 1, 29-106 (1999).
5. Deitmar, A. – Geometric zeta functions for locally symmetric spaces. To appear in: Amer. J. Math.
6. Deligne, P. – La conjecture de Weil. II. Publ. Math., IHES **52**, 137-252 (1980).
7. Godement, R.; Jacquet, H. – Zeta functions of simple algebras. Lecture Notes in Mathematics. 260. Berlin-Heidelberg-New York: Springer-Verlag. (1972).
8. Hashimoto, K. – On zeta and L-functions on finite graphs. Int. J. Math. **1**. no 4, 381-396 (1990).
9. Jacquet, H. – Principal L-functions of the linear group. Proc. Symp. Pure Math. **33**, pt 2, 63-86 (1979).
10. Manning, A. – Axiom A diffeomorphisms have rational zeta functions. Bull. Lond. Math. Soc. **3**, 215-220 (1971).
11. Margulis, G.A. – Discrete subgroups of semisimple Lie groups. Springer-Verlag, Berlin (1991).
12. Pierce, R.S. – Associative algebras. Springer-Verlag, New York-Berlin (1982).
13. Ruelle, D. – Zeta functions for Expanding maps and Anosov flows. Invent. math. **34**, 231-244 (1976).
14. Soulé, C. – On the zeroes of automorphic L-functions. Preprint.
15. Strassberg, H. – L-functions for $GL(n)$. Math. Ann. **245**, 23-36 (1979).
16. Tate, J. – Fourier Analysis in Number Fields and Hecke's Zeta-Functions. Algebraic Number Theory. Ed.: J. Cassels, A. Fröhlich, Thompson Book Company, Washington (1967).
17. Tate, J. – Number theoretic background. Automorphic forms, representations and L-functions, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. **33**, 2, 3-26 (1979).
18. Weil, A – Basic number theory. Classics in Mathematics. Springer-Verlag, Berlin (1995).

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